③ Infinitesimal Rotations : δa about a fixed axis.

→ Matrix representation

i) δa about $\hat{z}$-axis

$$\left[1-\frac{i}{\hbar}\delta a L_z\right]|x,y,z\rangle = \left[1-\frac{i}{\hbar}\delta a(\tilde{x}\tilde{p}_y-\tilde{y}\tilde{p}_x)\right]|x,y,z\rangle$$

$$= \left[1-\frac{i}{\hbar}\tilde{p}_x(-\delta a\cdot y)-\frac{i}{\hbar}\tilde{p}_y(\delta a x)\right]|x,y,z\rangle$$

$$= |x-\delta a\cdot y,\ y+\delta a\cdot x,\ z\rangle$$ ⇒ This is indeed the rot. $|R_{\hat{z}}(\delta a)\vec{x}\rangle$.

For $|\alpha\rangle$, an arbitrary ket of a spinless particle,

$$\langle x,y,z|\left[1-\frac{i}{\hbar}\delta a L_z\right]|\alpha\rangle = \langle x+\delta a y,\ y-\delta a x,\ z|\alpha\rangle \quad \cdots (*)$$

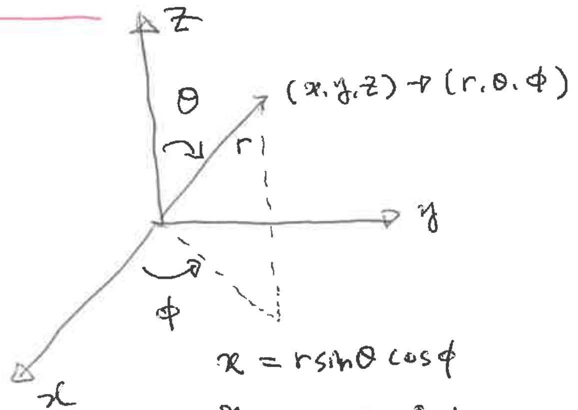
$$\hookrightarrow = \left[\left(1+\frac{i}{\hbar}\delta a L_z\right)|x,y,z\rangle\right]^{+}$$

In terms of a wave function,

$$\Psi_{R\alpha}(\vec{x}) = \Psi_{\alpha}(R^{-1}\vec{x}).$$

* Representation of L_z in the position space (spherical coordinates)

$(*) \Rightarrow \langle x,y,z|\alpha\rangle \longrightarrow \langle r,\theta,\phi|\alpha\rangle$



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

rotation

$\theta \to \theta+\delta\theta$, $\phi\to\phi+\delta\phi$

$\delta x = r\cos\theta\cos\phi\,\delta\theta - r\sin\theta\sin\phi\,\delta\phi$

$\delta y = r\cos\theta\sin\phi\,\delta\theta + r\sin\theta\cos\phi\,\delta\phi$

$\delta z = -r\sin\theta\,\delta\theta$.

Now, look at $\langle x+\delta a\, y,\ y-\delta a\, x,\ z|\alpha\rangle$.

$$\Rightarrow \quad \delta x = y\,\delta a, \quad \delta y = -x\,\delta a, \quad \delta z = 0.$$

$$\Rightarrow \quad \delta\theta = 0, \quad \delta\phi = -\delta a$$

Thus, $\langle x,y,z|\left[1-\frac{i}{\hbar}\delta a L_z\right]|\alpha\rangle$

$$= \langle x,y,z|\alpha\rangle - \delta a \frac{\partial}{\partial\phi}\langle x,y,z|\alpha\rangle + O(\delta a^2).$$

$$\Rightarrow \quad \langle\vec{x}|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle\vec{x}|\alpha\rangle$$

or $L_z \doteq -i\hbar\frac{\partial}{\partial\phi}$

ii) δa about $\hat{x}$-axis.

$$\langle\vec{x}|\left[1-\frac{i}{\hbar}\delta a L_x\right]|\alpha\rangle = \langle x,\ y+z\delta a,\ z-y\delta a|\alpha\rangle.$$

$$\Rightarrow \delta x = 0, \quad \delta y = z\,\delta a, \quad \delta z = -y\,\delta a \qquad \| \; L_x = \tilde{y}\tilde{p}_z - \tilde{z}\tilde{p}_y$$

$$\Rightarrow \quad \delta\theta = \frac{y}{r\sin\theta}\delta a = \sin\phi\,\delta a.$$

$$\delta\phi = \frac{1}{r\sin\theta\cos\phi}\left[\delta y - r\cos\theta\sin\phi\,\delta\theta\right]$$

$$= \frac{1}{r\sin\theta\cos\phi}\left[r\cos\theta - r\cos\theta\sin^2\phi\right]\delta a$$

$$= \cot\theta\cos\phi\,\delta a$$

Thus, $\langle\vec{x}|\left[1-\frac{i}{\hbar}\delta a L_x\right]|\alpha\rangle$

$$\cong \langle\vec{x}|\alpha\rangle + \delta\theta\frac{\partial}{\partial\theta}\langle\vec{x}|\alpha\rangle + \delta\phi\frac{\partial}{\partial\phi}\langle\vec{x}|\alpha\rangle$$

$$\Rightarrow \quad \langle\vec{x}|L_x|\alpha\rangle = -i\hbar\left[-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]\langle\vec{x}|\alpha\rangle.$$

or $L_x \doteq -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$

iii) $L_y = \frac{1}{i\hbar}[L_z, L_x]$

$$\doteq -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

iv)
$$L_\pm = L_x \pm iL_y$$
$$\doteq -i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)$$

v) $\vec{L}^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+)$

$$\doteq -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]$$

③ a particle in a central potential.

$$H = \frac{\vec{\underset{\sim}{P}}^2}{2m} + V(\vec{\underset{\sim}{X}}) \quad \| \; V(\vec{\underset{\sim}{X}})|\vec{x}\rangle = V(r)|\vec{x}\rangle.$$

central potential.

→ invariant under rotations

$$[H, \vec{L}] = 0 \quad , \quad [H, \vec{L}^2] = 0$$

→ (l, m) are good quantum numbers !

* NOTE :

$$[L_i, \vec{u}\cdot\vec{v}] = 0 \quad \xrightarrow{\text{ex.)}} \quad [L_i, \vec{P}^2] = 0.$$

$$[L_i, (\vec{u}\times\vec{v})_j] = i\hbar\,\varepsilon_{ijk}(\vec{U}\times\vec{V})_k$$

for $\vec{u}$ an $\vec{v}$ under rotations.

* recall $[J_i, S] = 0$ $\quad \| \; S$: scalar operator

$$[J_i, V_j] = i\hbar\,\varepsilon_{ijk}V_k \quad , \quad \text{or} \quad \vec{J}\times\vec{V} = i\hbar\vec{V}$$

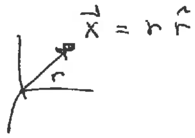
also, $\vec{L}^2 = \vec{x}^2 \vec{p}^2 - (\vec{x}\cdot\vec{p})^2 + i\hbar\, \vec{x}\cdot\vec{p}$

To prove, use $(\vec{a}\times\vec{b})^2 = \vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b}) + \gamma\, \vec{a}\cdot\vec{b}$ when $\begin{cases} [a_i, b_j] = \gamma\delta_{ij} \\ [b_i, b_j] = 0 \end{cases}$

or see (3.6.17) of Sakurai & Napolitano

$$\Rightarrow \langle \vec{x}|\vec{L}^2|\alpha\rangle = \langle \vec{x}|\vec{x}^2\vec{p}^2|\alpha\rangle - \langle \vec{x}|(\vec{x}\cdot\vec{p})^2|\alpha\rangle + i\hbar\langle \vec{x}|\vec{x}\cdot\vec{p}|\alpha\rangle$$

In the Spherical coordinates,



$$\langle \vec{x}|\vec{x}\cdot\vec{p}|\alpha\rangle = -i\hbar\, r\frac{\partial}{\partial r}\langle \vec{x}|\alpha\rangle$$

$$\langle \vec{x}|(\vec{x}\cdot\vec{p})^2|\alpha\rangle = -\hbar^2\, r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\langle \vec{x}|\alpha\rangle\right)$$

$$= -\hbar^2\left[r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}\right]\langle \vec{x}|\alpha\rangle$$

$$\langle \vec{x}|\vec{x}^2\vec{p}^2|\alpha\rangle = r^2\langle \vec{x}|\vec{p}^2|\alpha\rangle.$$

$\Rightarrow$ Schrödinger eq.

$$\langle \vec{x}|H|\alpha\rangle = E\langle \vec{x}|\alpha\rangle$$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]\langle \vec{x}|\alpha\rangle + \frac{1}{2mr^2}\langle \vec{x}|\vec{L}^2|\alpha\rangle = E\langle \vec{x}|\alpha\rangle$$

$$+ V(r)\langle \vec{x}|\alpha\rangle$$

Since l, m are good quantum numbers,

$$|\alpha\rangle \equiv |\underset{\uparrow}{n}, \underset{\uparrow \vec{L}^2}{l}, \underset{\uparrow L_z}{m}\rangle.$$

↑ for the radial part

$$\Rightarrow -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right]\langle \vec{x}|n,l,m\rangle$$

$$= E\langle \vec{x}|n,l,m\rangle$$

Schrodinger eq.

→ Radial Equations

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+V_{eff}^{(\ell)}(r)\right]R_{n\ell}(r) = E\,R_{n\ell}(r)$$

* Are "n" and "ℓ" enough for $R(r)$?

i) ℓ : obvious ← $V_{eff}^{(\ell)}$

ii) n : "Sturm-Liouville" theory : bound states are non-deg. and Real; (See also HW#5.1) in 1D.

Thus, $\langle \vec{x} | n, \ell, m\rangle \equiv R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$

↑ radial eq. ↑ eigenfunction of $\vec{L}^2$ and L_z.

④ Spherical Harmonics : $Y_\ell^m(\theta,\phi) = \langle \hat{n} | \ell, m\rangle$ ‖ $\hat{n} = \frac{\vec{x}}{|\vec{x}|}$

$$L_z |\ell,m\rangle = m\hbar\, |\ell,m\rangle \quad \cdots\cdots (*)$$

$$\vec{L}^2 |\ell,m\rangle = \ell(\ell+1)\hbar^2 |\ell,m\rangle \quad \cdots\cdots (**)$$

i) $\langle \hat{n}| \cdot (*)$: $-i\hbar\frac{\partial}{\partial\phi} Y_\ell^m(\theta,\phi) = m\hbar\, Y_\ell^m(\theta,\phi)$

→ $Y_\ell^m(\theta,\phi) \propto \exp[im\phi]$.

∴ Integer m's are only allowed!

Here, we're talking about "spatial" wave-functions.

→ $\psi(r,\theta,0) = \psi(r,\theta,2\pi)$ to be single-valued in "position space"

☆☆☆ m = integers : $-\ell, -\ell+1, \cdots \ell-1, \ell$.

So, ℓ = integers. ☆☆☆ for the "orbital" angular momentum.